arXiv:gr-qc/9911110v3 17 Nov 2000

On the birth of a closed hyperbolic universe

S. S. e Costa and H. V. Fagundes

Instituto de Física Teórica, Universidade Estadual Paulista,

Rua Pamplona, 145, São Paulo, SP 01405-900, Brazil,

E-mail: sancosta@ift.unesp.br, helio@ift.unesp.br

February 7, 2008

Abstract

We clarify and develop the results of a previous paper on the birth of a closed universe of negative spatial curvature and multiply connected topology. In particular we discuss the initial instanton and the second topology change in more detail. This is followed by a short discussion of the results.

PACS numbers: 98.80.Hw, 04.20.Gz

1 Introduction

In a recent paper [1] we[1] suggested a process for the spontaneous creation of a universe with closed - i. e., compact and boundless - spatial sections of negative curvature. (A short report on the same subject was presented at the *Cosmological Topology in Paris 1998* meeting [2].) This process involved four steps: (i) the actualization of an instanton of nontrivial global topology into a de Sitter universe of positive spatial curvature; (ii) a topology and metric change into a closed de Sitter world of negative spatial curvature; (iii) inflation of this hyperbolic de Sitter universe; and (iv) reheating and beginning of the radiation era with the metric of Friedmann's open model ($\Omega_0 < 1,\ \Lambda = 0$) and the spatially compact topology obtained in step (ii). In Sections 2 and 3 we justify and develop steps (i) and (ii) in more detail. Steps (iii) and (iv) may be taken as the same as in the usual inflationary scenarios - see [3], Chapter 8, for example. The last section briefly argues for the compatibility of a compact hyperbolic universe both with the observed fluctuations of the cosmic microwave background (CMB) and with an inflationary scenario leading to a present density ratio $\Omega_0 < 1$.

[1] In Ref. [1] the first author appeared by mistake with name S. S. da Costa. His correct name is S. S. e Costa, as above.

2 The instanton orbifold

We modeled the spontaneous birth in Vilenkin [4]. But while he has an S^4 instanton tunneling into an $R \times S^3$ spherical universe (where S^n is the n-sphere and R is the time axis), we start from a more complex structure in order to reach a spherical spacetime $\mathcal{M}_L = R \times (S^3/\Gamma)$ with nontrivial topology. Here $M = S^3/\Gamma$ is the quotient space of S^3 by a discrete, finite group of isometries Γ, which acts freely on S^3; cf. [5], for example. If S^3 has unit radius the volume of M is $2\pi^2/|\Gamma|$, where $|\Gamma|$ is the number of elements of Γ, so we have a variety of spherical manifolds that may, in principle, be chosen as spatial sections of positive curvature for a Robertson-Walker model. In the example of [1] M is the lens space $L(50,1)$, with volume $2\pi^2/50$.

Instead of S^4 we construct a more general instanton S^4/Γ, which we proceed to describe. The action of Γ on $S^4 = \{(X_\alpha, \alpha = 0{-}4) \in R^5;\ X_\alpha X_\alpha = 1\}$ is obtained by extending its action on the standard (unit radius) S^3 to all 'parallel' 3-spheres on S^4, that is, for $|X_0| \leq 1$, $S^3_{X_0} = \{(X_0,\ X_i, i = 1-4);\ X_i X_i = 1 - X_0^2\}$. The action is already defined on the 'equator' S^3_0, which is isometric to S^3. Let $(X_0, X_i) \in S^3_{X_0}$ and $\gamma \in \Gamma$. If $|X_0| < 1$, then $(0,\ X'_i = X_i/\sqrt{1 - X_0^2}) \in S^3_0$, so that $\gamma(0, X'_i) = (0, X''_i) \in S^3_0$, and we define $\gamma(X_0, X_i) \equiv (X_0,\ X''_i\sqrt{1 - X_0^2}) \in S^3_{X_0}$. If $|X_0| = 1$, then $\gamma S^3_{\pm 1} = S^3_{\pm 1}$,

which are the poles of S^4. Thus the action of Γ on S^4 is not free, and so the quotient space S^4/Γ is not a manifold, but an orbifold with two cone points corresponding to the poles of S^4 - cf. Scott [5], Sec. 2.

Actually only the lower half ($X_0 \leq 0$) of the instanton takes part in the solution. Following Gibbons [6] we call this manifold $\mathcal{M}_R$ - the index R meaning Riemannian (the positive definite part of the solution, popularly known as Euclidean on account of the metric signature). The full spacetime solution is $\mathcal{M} = \mathcal{M}_R \cup_\Sigma \mathcal{M}_L$, where $\mathcal{M}_R$ and $\mathcal{M}_L$ are attached smoothly by $\Sigma = S_0^3/\Gamma = \partial\mathcal{M}_R$. With this generalization Gibbons's conditions are satisfied: $\mathcal{M}_R$ is a compact orbifold with Σ as sole boundary; Σ is a Cauchy surface for $\mathcal{M}_L$; and it has a vanishing second fundamental form with respect to both $\mathcal{M}_R$ and $\mathcal{M}_L$ - this is true of the S^3 covering, and the action of Γ does not interfere with the local metrics.

3 The second topology change

As described in [1] the first epoch after creation had the metric

$$ds^2 = -dt^2 + r_0^2 \cosh^2(t/r_0)(d\chi^2 + \sin^2\chi \; d\Omega^2) \; , \qquad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\ d\varphi^2$ and r_0 is Planck's length or time; and the topology $R \times M$ discussed in the preceding section. Then we assumed a formalism developed by De Lorenci et al. ([7]; hereafter LMPS) could be used to justify a quantum transition into a second epoch with topology $R \times M'$, where M' is a compact hyperbolic manifold, and metric

$$ds^2 = -d\tau^2 + r_0^2 \sinh^2(\tau/r_0)(d{\chi'}^2 + \sinh^2\chi'\ d\Omega^2)\ , \tag{2}$$

In the example of [1] M' is Weeks manifold, which is the smallest space in the *SnapPea* census [8].

To match these two stages we postulated conservation of physical volume. But in order to use the results in LMPS we should rather have continuity of the expansion factor: if t_f is the final time of stage one and τ_i is the initial time of stage two, then this continuity requires $\cosh(t_f/r_0) = \sinh(\tau_i/r_0)$. The homogenizing process to be produced by inflation in stage two demanded that τ_i was of the order of Planck's time r_0. To keep a number from the example in [1], let $\tau_i = 0.9865 r_0$. If follows that $t_f/r_0 = 0.5489$. In that example this time interval would not allow for the homogenization of space M. However, this first stage is so short that it may eventually, in a complete theory, be viewed as a quantum intermediate state. Anyway, it probably does not make sense to speak of density smoothening in a sub-Planckian scale. As

for the universe's homogenization, it is taken care of by the 70-odd e-fold inflation of our second epoch, as in more usual scenarios.

Now we proceed to give estimates of the probabilities for the topology change between these stages, according to LMPS. It would be desirable to obtain absolute probabilities, but in the present stage this is not possible, because their wave functions are not normalized. LPMS calculate conditional probabilities for transitions among three topologies on manifolds M_k, one for each sign of the curvature, $k = 0, \pm 1$. Here we shall restrict ourselves to M and M'; the case for a Euclidean manifold M_0 is unclear, given the arbitrariness and continuous range of its fundamental polyhedron's volume.

We need an additional hypothesis in order to apply LMPS's results. The latter assumes null potentials $U(\phi) = V(\xi) = 0$, but since these potentials enter their Hamilton-Jacobi equation only in the combination $U(\phi) + V(\xi)$, the same equation is obtained by only requiring $U(\phi) = -V(\xi) > 0$. Although this condition looks contrived, we need it at present because our transition in [1] was supposed to take place near the false vacuum.

In LMPS the calculations hinge on functions F_k, which we rewrite, in Planckian units, as

$$F_k(M_k) = \frac{\bar{a}}{2\pi m} \int_{M_k} \cos(2\sqrt{k}\chi) \sin\theta \, d\chi \, d\theta \, d\varphi \ , \tag{3}$$

where $\bar{a}$ is the expansion factor at the moment of the transition, and m is the mass associated with an auxiliary field ξ, which "is introduced to give a notion of time evolution to the quantum states." (This field is their version of Kuchăr and Torre's [9] "reference fluid.")

The last equation gives immediately $F_1(M) = 0$, because for the lens space the range of χ is $[0, \pi]$ for any values of θ and φ.

For $k = -1$ Eq. (72) in LMPS turned out to be impractical for actual evaluation; only lower and upper bounds were obtained for their $F_{-1}(I^3)$. We succeeded in performing the integration in our case by first expressing Eq. (3) in hyperbolic cylindrical coordinates (ρ, φ, z), which are related to the spherical coordinates (χ, θ, φ) by $\sinh\rho = \sinh\chi \sin\theta$, $\tanh z = \tanh\chi \cos\theta$, and $\cosh\chi = \cosh\rho \cosh z$. Then we get

$$F_{-1}(M') = \frac{\bar{a}}{2\pi m} \left[2V(M') + \int_{M'} \frac{\sinh\rho \cosh\rho \ d\rho \ d\varphi \ dz}{\cosh^2\rho \cosh^2 z - 1} \right] \ , \tag{4}$$

where $V(M') = 0.942707$ is the volume of Weeks manifold. The integral was calculated by decomposing the fundamental polyhedron for M' into quadrirectangular tetrahedra, and using results of hyperbolic geometry as given by

Coxeter [10] and Coolidge [11]. This computation was carried out by one of us (SSC) , and is discussed elsewhere [12]. The result is $F_{-1}(M') = 1.4777\, \bar{a}/m$.

Let the wave function of the universe be $\Psi(a, \phi, \xi, M_k)$, where a is the expansion factor and ϕ is the inflaton field. Similarly to LMPS we put $|\Psi(\bar{a}, \bar{\phi}, \xi, M')|^2 = A(\bar{a}, \bar{\phi}) \exp(2F_{-1}\xi)$, $\quad |\Psi(\bar{a}, \bar{\phi}, \xi, M)|^2 = C(\bar{a}, \bar{\phi}) \exp(2F_1\xi)$, where A and C are positive functions. Then the ratio of probabilities that the universe is found with spaces M' and M at "time" ξ is $P(M')/P(M) = (A/C) \exp(2.9554\, \bar{a}\xi/m)$. This is null for $\xi = -\infty$, which implies initial space M, and infinite for $\xi = +\infty$, hence final state M'. Thus we get the desired topology change.

We are aware that LMPS's formalism suffers from the usual doubts and limitations of quantum cosmology calculations. But we hope it is a step in the right direction.

4 Discussion

Recently the theoretical preference for flat space cosmology has been reinforced by observations - see, e. g., [13] and references there - that suggest a substantial present value of the cosmological constant Λ, making up a total

critical density: $\Omega_0 = \Omega_{matter} + \Omega_\Lambda = 1$. But this belief is not universal - cf. [14], for example; should it become untenable, we may have to face a subcritical density and a universe with negative spatial curvature. There is even the possibillity of $\Omega_0 < 1$ in the presence of a positive Ω_Λ; cf. Quast and Helbig [15] and references there. Recent observational results, as quoted by Lehoucq et al. [16], only restricts Ω_0 to the range $[0.88, 1.12]$.

It has been argued [17] that the CMB fluctuations are incompatible with a closed hyperbolic model (with $\Lambda = 0$) unless $\Omega_0 \approx 1$, and its spatial dimensions are of the order of magnitude of the observable universe. The recent work of Aurich [18] seems to contradict this. See also Inoue et al. [19], Cornish and Spergel [20]. But even if Bond et al. [17] are correct, the case for a closed hyperbolic universe still deserves investigation. And since it might not be small enough [21] to account for the homogeneity of cosmic images (the substitute for the true homogeneity of simply connected models), we should be prepared to associate compactness with inflation, as discussed in [21] and done here. The usual inflationary scenario tends to exclude the open Friedmann model on the grounds of a needed fine-tuning of the density ratio $\Omega(t)$ in early times. Thus at the beginning of the radiation epoch in our model, $t_1 = 71 t_{Planck}$, the equations in [3], Chapter 3, indicate

$\Omega(t_1) \approx 1 - 1 \times 10^{-57}$, which looks suspicious for the open model. However, if we find that creation and early evolution were governed by topological constraints, then the fact of a pre-inflationary negative curvature being diluted by inflation could only lead to a value of $\Omega(t_1)$ that was very close to, but still smaller than one. This is so because the by then frozen topology on a compact 3-space could not support a Euclidean metric - cf. [22]. (A similar argument has been made by Padmanabhan [23], but it does seem to hold in his context of infinite spatial sections.)

One of us (SSeC) thanks Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP - Brazil) for a doctorate scholarship.

References

[1] e Costa, S. S. and Fagundes, H. V. 1999. *Gen. Rel. Grav.* **31**, 863

[2] e Costa, S. S. and Fagundes, H. V. 1999. In *E-Proceedings of Cosmological Topology in Paris 1998* meeting, eds. V. Blanloeil and B. F. Roukema, **www.iap.fr/users/roukema/CTP98**

[3] Kolb, E. W. and Turner, M. S. 1990. *The Early Universe* (New York, Addison-Wesley)

[4] Vilenkin, A. 1982. *Phys. Lett.* **117B**, 25

[5] Scott, P. 1983. *Bull. London Math. Soc.* **15**, 401

[6] Gibbons, G. W. 1998. *Class. Quantum Grav.* **15**, 2605

[7] De Lorenci, V. A., Martin, J., Pinto-Neto, N., and Soares, I. D. 1997. *Phys. Rev.* **D56** 3329

[8] Weeks, J. R. *SnapPea: A Computer Program for Creating and Studying Hyperbolic Manifolds*, available at Web site **thames.northnet.org/weeks**

[9] Kuchǎr, K. V. and Torre, C. G. 1991 *Phys. Rev.* **D43** , 419

[10] Coxeter, H. S. M. 1965. *Non-Euclidean Geometry*, 5th edition (University of Toronto Press)

[11] Coolidge, J. L. 1927. *The Elements of Non-Euclidean Geometry* (Oxford University Press)

[12] e Costa, S. S. 2000. *Phys. Rev.* **D62**(4), 047303

[13] Chiba, M. and Yoshii, Y. 1999. *Astrophys. J.* **510** , 42

[14] Aguirre, A. N. 1999. *Astrophys. J.* (*Letters*) **512**, L19

[15] Quast, R. and Helbig, H. 1999. Preprint astro-ph/9904174

[16] Lehoucq, R., Uzan, J.-P., and Luminet, J.-P., 2000. Preprint astro-ph/0005515

[17] Bond, J. R., Pogosyan, D. and Souradeep, T. 1998. In *Proc. 17th Texas Symp. Relat. Astrophysics,* eds. A. Olinto, J. A. Frieman, and D. N. Schramm (World Scientific, Singapore)

[18] Aurich, R. 1999. Preprint astro-ph/9903032

[19] Inoue, K. T., Tamita, K., and Sugiyama, N. 1999. Preprint astro-ph/9906304

[20] Cornish, N. J. and Spergel, D. N. 1999. astro-ph/9906401

[21] Ellis, G. F. R. and Schreiber. G. 1986. *Phys. Lett.* **115A**, 97

[22] Thurston, W. P. 1982. *Bull. Am. Math. Soc.* **6** , 357

[23] Padmanabhan, T. 1993. *Structure formation in the universe* (Cambridge U. Press), Chapter 10